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On the Univariate Representation of BEKK Models with Common Factors

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Abstract

First, we investigate the minimal order univariate representation of some well known $n-$dimensional conditional volatility models. Even simple low order systems (e.g. a multivariate GARCH(0,1)) for the joint behavior of several variables imply individual processes with a lot of persistence in the form of high order lags. However, we show that in the presence of common GARCH factors, parsimonious univariate representations (e.g. GARCH(1,1)) can result from large multivariate models generating the conditional variances and conditional covariances/correlations. The trivial diagonal model without any contagion effects in conditional volatilities gives rise to the same conclusions though.

Consequently, we then propose an approach to detect the presence of these commonalities in multivariate GARCH process. The factor we extract is the volatility of a portfolio made up by the original assets whose weights are determined by the reduced rank analysis.

We compare the small sample performances of two strategies. First, extending Engle and Marcucci (2006), we use reduced rank regressions in a multivariate system for squared returns and cross-returns. Second we investigate a likelihood ratio approach, where under the null the matrix parameters of the BEKK have a reduced rank structure (Lin, 1992). It emerged that the latter approach has quite good properties enabling us to discriminate between a system with seemingly unrelated assets (e.g. a diagonal model) and a model with few common sources of volatility.

JEL: C10, C32

Keywords: Common GARCH, factor models, BEKK, Final equations.

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1 Motivation

This paper proposes an alternative view to interpret and investigate the presence of common volatility in multivariate asset return models. Indeed, we start with the observation that parsimonious univariate volatility models (e.g. GARCH(1,1)) are often detected in empirical works. We show that this can be seen as an indication of the presence of few factors generating the volatility of asset returns. We get this result using the final equation representation of multivariate models, tools ushered from Zellner and Palm (see *inter alia* 1974, 1975, 2004) for VARMA and extended to reduced rank models by Cubadda *et al.* (2009). This framework allows us to derive the marginal GARCH representation for the conditional variances and conditional covariances of the multivariate GARCH model (e.g. Nijman and Sentana (1996) in the unrestricted case) but in the presence of common GARCH factors.

Given that both independent volatility models (e.g. a diagonal model) and a highly dependent structure with few factors generating the conditional second moments deliver similar parsimonious marginal representations, we propose a new framework for detecting the presence of common volatility. Our setting is different from two types of modeling commonly found in this literature. First, contrary to the usual factor models in finance, our approach does not assume any ad hoc idiosyncratic component to be added to the common factor representation. Second, with the volatility being unobservable, we do not directly test for potential combinations of hereroskedastic series that are conditionally homoskedastic (see Engle and Susmel, 1993; Engle and Kozicki 1993). Instead we translate the task to detect the presence of common volatility in asset returns into an analysis of common cyclical features (Vahid and Engle, 1993) in the dynamics of the squared returns and cross-returns. To some extent our framework can be seen as a generalization of the Engle and Marcucci (2006) pure variance model although they do not consider the covariances in their analysis. Moreover, they tailor the dynamics of the logarithms of squared returns to exactly match their theoretical assumptions and they do not consider the presence of non-i.i.d. disturbances.

The common volatility factors extracted by our strategy are quite intuitive since they represent the conditional variance of a portfolio composed by the series involved in the analysis. Applying our common cycle approach and hence testing for the presence of commonalities in the *vech* or the *vecd* of the conditional covariance matrix is hazardous however. Indeed, even under the null, the combination that annihilates the temporal dependence in squared returns and cross-returns is a martingale difference sequence. We show that this leads to large size distortions if one uses canonical correlation test statistics as in Engle and Marcucci (2006). Moreover, accounting for the presence of hereroskedasticity using a robustified version of those reduced rank tests (Hecq and Issler, 2012) does not help to get rid of the effect of non-normality. It turns out that, although the representation of multivariate systems in terms of squared returns and cross-returns is crucial to obtain the orders of the marginal models, only a maximum likelihood strategy consisting in estimating a multivariate GARCH for the returns under the reduced rank null hypothesis enables us to draw some conclusions. Also note that our goal is not to provide a glossary of marginal representations obtained from every multivariate systems[1] but to provide tools to understand the underlying behavior of a parsimonious block of assets. As an example, among the bunch of fifty daily stock returns that we analyze in this paper, six of them are rather better described as GARCH(1,1) than as long memory processes. Then we will investigate whether there is a factor structure behind this feature or simply an absence of causality from the past covariances to the variances.

The rest of this paper is as follows. Section 2 sets up the notations and derives the general results for the final equation representation of multivariate GARCH models. We propose in Section 3 some multivariate models accounting for co-movements in volatility and show that the implied marginal volatility processes are of low order. We introduce the pure portfolio common GARCH volatility model based on the factor GARCH specification proposed by Lin (1992) for the BEKK. Section 4 extends the previous results to the multiple factor cases. This will be important for our approach because the dimension of the system for $k > 1$ factors is not k but some $\tilde{k} > k$. We develop in Section 5 several testing strategies that we evaluate in Section 6 using a Monte Carlo exercise. In Section 7 we apply our preferred test to six US stock return series. In particular we are able to discriminate between a diagonal model and a general multivariate framework, with correlated conditional variances and contagion effects, driven by a small number of common factors in volatility. The final section concludes.

2 The final equation representation of multivariate GARCH models

2.1 Definition of common volatility features

To state the notation for univariate excess returns[2], ε_{1t} is such that $\varepsilon_{1t} = u_{1t}\sqrt{h_{11t}}$, where u_{1t} has any centered parametric distribution with unitary variance and the conditional variance of ε_{1t} follows a GARCH(p, q) with $h_{11t} = \omega_1 + \sum_{j=1}^{p} \beta_{1,j} h_{11t-j} + \sum_{i=1}^{q} \alpha_{1,i}\varepsilon_{1t-i}^2$. Consequently here, p refers to the GARCH terms and q is the order of the moving average ARCH term. The error in the squared returns is obtained as usual using $v_{1t} = \varepsilon_{1t}^2 - h_{11t}$. As an example a GARCH$(1, 2)$ can be rearranged for the squared returns such that $\varepsilon_{1t}^2 = \omega_1 + (\alpha_{1,1} + \beta_{1,1})\varepsilon_{1t-1}^2 + \alpha_{1,2}\varepsilon_{1t-2}^2 + v_{1t} - \beta_{1,1}v_{1t-1}$ (see *inter alia* Bollerslev,

[1] Let us just name the diagonal model, the constant conditional correlation (CCC), the dynamic conditional correlation (DCC), the dynamic equicorrelation (DECO, see Engle and Kelly, 2008), the approach by Baba, Engle, Kraft and Kroner (1999, hereafter BEKK), the orthogonal GARCH, the factor GARCH, as well as some of their block versions such as the BLOCK-DCC (Billio, Caporin and Gobbo, 2003) and BLOCK-DECO (Engle and Kelly, 2008) as examples proposed in the literature to restrict the multivariate setting towards a manageable size as well as to impose the positive definitiveness of the covariance matrix (see *inter alia* the surveys by Bauwens, Laurent and Rombouts, 2006 and Silvennoinen and Terasvirta, 2009).

[2] By excess return we mean that the conditional mean (a constant or an ARMA model for instance) has been substracted from the returns r_t, say $\varepsilon_t = r_t - E(r_t|\Omega_{t-1})$, where Ω_{t-1} denotes the information set up to and including $t - 1$.

1996). Thus, the squared returns follow a heteroskedastic weak GARCH univariate ARMA$(2,1)$ process. In general a GARCH(p,q) has got an ARMA$(\max(p,q),p)$ representation for the squared returns. In most of the examples we have considered, we take $q \geq p$.

For multivariate modeling, we denote by $\varepsilon_t = H_t^{1/2} z_t$, $t = 1, \ldots, T$, the $n \times 1$ vector of excess returns of financial assets observed at the time period t. T is the number of observations and $z_t \sim i.i.d.(0, I_n)$. Consequently we have $\varepsilon_t | \Omega_{t-1} \sim D(0, H_t)$ with Ω_{t-1} the past information set and H_t the conditional covariance matrix of the n assets that is measurable with respect to Ω_{t-1}; $D(.,.)$ is some arbitrary multivariate distribution. Let us also denote without loss of generality $H_t = H + \tilde{H}_t$ as the sum of a constant part and a time-varying part.

In this multivariate setting, testing for common GARCH (in the sense of Engle and Kozicki, 1993) amounts to look towards s directions $\delta' \varepsilon_t$ such that the $\delta' \varepsilon_t$ combinations of asset returns are conditionally homoskedastic, namely such that $\delta' \varepsilon_t | \Omega_{t-1} \sim D(0, C)$ where C does not depend on t. To be more explicit let us define the following partitioned matrix Λ spanning $\mathcal{R}^n$

$$\Lambda = \begin{pmatrix} I_s & \delta^{*\prime}_{s \times (n-s)} \\ 0_{(n-s) \times s} & I_{n-s} \end{pmatrix},$$

where we have normalized the cofeature space $\delta' = (I_s : \delta^{*\prime}_{s \times (n-s)})$ for the sake of notation. We define a common volatility feature model iff for

$$\Lambda \varepsilon_t | \Omega_{t-1} \sim D(0, \Lambda H_t \Lambda'),$$

and hence in

$$\Lambda \varepsilon_t | \Omega_{t-1} \sim D(0, \Lambda H \Lambda' + \Lambda \tilde{H}_t \Lambda'),$$

we have

$$rank(\Lambda \tilde{H}_t \Lambda') = n - s$$

instead of n. Clearly this definition splits in a natural way, i.e. without referring to an ad hoc factor structure, the volatility feature of the $n-$dimensional process as the sum of two parts. There exists s combinations of the asset returns that have a common volatility component. Consequently there exist s linear combinations of ε_t that have time-invariant conditional distributions. The remaining $n - s$ combinations generate the time-varying volatility of the system.

This approach, although not formulated in this way can be seen as a generalization of Engle and Kozicki (1993) and Engle and Susmel (1993). Indeed, it is a generalization because both papers only look at bivariate processes (i.e. $n = 2$) with potentially the presence of a single factor, namely $k = n - s = 1$. Our paper goes beyond these aforementioned papers because we propose a multivariate framework for $n \geq 2$ and $k \geq 1$. The limitations of the previous papers come also from the fact that a grid search is used to find a combination of series such that this combination minimizes the LM test of no ARCH. Not only is this strategy computationally inefficient but it is tedious to extend to more than two series. This explains why bivariate analyses are usually considered in Ruiz (2009) or in Arshanapalli *et al.* (1997) for instance. Note also that our approach might look like a factor model in volatility with a constant idiosyncratic component, namely when only the factor generates time-varying volatility. This is very different to the usual approach proposed in the literature (e.g. King and Wadhwani, 1990) in which the distribution of the idiosyncratic part can also be time-varying. In this sense we are closer to Diebold and Nerlove (1989) although in our framework the number of factors is going to be tested. Finally, our approach improves upon Engle and Marcucci (2006) in three respects. First we investigate the theoretical impact of the addition of the cross-products to their pure variance model. Second, we do not assume a particular form such as an exponential model. Third we evaluate different testing strategies in a Monte Carlo experiment.

2.2 Final equation results

Similarly to the univariate case, and using half-vectorization operators for the $vech(H_t)$, let us denote $v_t = vech(\varepsilon_t \varepsilon_t^{'}) - vech(H_t)$. Therefore, a multivariate GARCH(0,1) (MGARCH(0,1) hereafter) can be written as a VAR(1) for observed squared returns and covariances. For instance a bivariate MGARCH(0,1) with $vech(H_t) = \{h_{11t}, h_{12t}, h_{22t}\}'$ and $vech(\varepsilon_t \varepsilon_t^{'}) = \{\varepsilon_{1t}^2, \varepsilon_{1t}\varepsilon_{2t}, \varepsilon_{2t}^2\}'$ gives

$$\begin{pmatrix} \varepsilon_{1t}^2 \\ \varepsilon_{1t}\varepsilon_{2t} \\ \varepsilon_{2t}^2 \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} + \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \begin{pmatrix} \varepsilon_{1t-1}^2 \\ \varepsilon_{1t-1}\varepsilon_{2t-1} \\ \varepsilon_{2t-1}^2 \end{pmatrix} + \begin{pmatrix} v_{1t} \\ v_{2t} \\ v_{3t} \end{pmatrix}, \tag{1}$$

where usual non-negativity and stationarity restrictions on the parameters are assumed to be met. Let us now generalize the system (1) for n assets and q lags, with $vech(\varepsilon_t \varepsilon_t^{'}) = \epsilon_t$ a $N = n(n+1)/2$ dimensional vector, such that

$$A(L)\epsilon_t = \omega + v_t, \tag{2}$$

with the lag operator L such that $Lz_t = z_{t-1}$. $A(L) = (I - A_1 L - ... - A_q L^q)$ is a matrix polynomial of degree q in this example, where A_i are coefficient matrices; ω and v_t are N-dimensional vectors. Note that (2) has the Wold representation $\epsilon_t = A^{-1}(L)\omega + A^{-1}(L)v_t$ of squared disturbances and cross products with v_t being a martingale difference stationary process as $E(v_t|\Omega_{t-1}) = 0$, $E(v_t v_{t-i}'|\Omega_{t-i}) = 0$, $i > 0$ and Ω_{t-i} being the past of v_t up to and including period $t - i$.

Let us now premultiply both sides of (2) by the adjoint of the matrix polynomial $A(L)$ to obtain

$$|A(L)|\epsilon_t = \omega^* + Adj\{A(L)\}v_t \tag{3}$$

where $|A(L)| = \det\{A(L)\}$, i.e. the determinant of the matrix polynomial $A(L)$, is a scalar polynomial in L; $Adj\{A(L)\}$ denotes the adjoint (or the adjugate) of $A(L)$ and $\omega^* = Adj\{A(L)\}\omega$. This can be rewritten as a system of univariate weakly ARMA models with autoregressive polynomial $|A(L)|$ and scalar moving average polynomial $\theta_i(L)$ and a white noise disturbance ζ_{it}

$$|A(L)|\epsilon_{it} = \omega_i^* + \theta_i(L)\zeta_{it}, \quad i = 1, ..., n. \tag{4}$$

Notice that each of the variables ζ_{it} is serially uncorrelated, but they are cross-correlated at different lags. In a MGARCH$(0, q)$, each component of the vector ϵ_t has a weak ARMA$(Nq, (N-1)q)$ or Wold representation that can be written as follows for instance for the first element ε_{1t}^2

$$|A(L)|\varepsilon_{1t}^2 = \omega_1^* + \theta_1(L)\zeta_{1t}. \tag{5}$$

Proposition 1 (see also Nijman and Sentana, 1996) summarizes the main features of the final equation representation (5).

Proposition 1 *In a $n-$dimensional MGARCH$(0, q)$, each univariate component is weakly GARCH with a univariate ARMA$(Nq, (N-1)q)$ representation of the squared returns and cross-returns with the same values for the autoregressive parameters, where $N = \frac{n(n+1)}{2}$. Consequently each component follows a weak GARCH$((N-1)q, Nq)$. The orders should be taken as upperbounds for the orders of the univariate ARMA and GARCH models.*

Proof. The proof is obvious from the definition of the determinant and the adjoint. This well known result is simply due to the fact that in the MGARCH$(0, q)$ for instance $|A(L)|$ contains by construction up to L^{Nq} terms and the adjoint matrix is a collection of $\{(N-1) \times (N-1)\}$ cofactor matrices, each of the matrix elements can contain the terms $1, L, ..., L^q$. As υ_t is a vector martingale difference sequence it is serially uncorrelated and each element of $Adj\{A(L)\}\upsilon_t$ can be represented as a univariate moving average and therefore it is a weak GARCH process. ■

Proposition 2 *Proposition 1 generalizes in a straightforward manner to show that for a $n-$dimensional MGARCH(p, q), squared and cross-returns have a univariate ARMA$(N \max\{p, q\}, (N-1)\max\{p, q\} + p)$ representation at most with the same values for the autoregressive parameters. Consequently each component follows a weak GARCH$((N-1)\max\{p, q\} + p, N\max\{p, q\})$ process at most.*

Proof. It is similar to that of Proposition 1. ■

The above outcomes, that apply the usual results of the VAR(p) and VARMA(p, q) are generally not in agreement with empirical findings suggesting low order univariate GARCH schemes. Indeed, for $n = 20$ assets, a MGARCH$(0, 2)$ implies individual ARMA$(420, 418)$ models in squared returns and cross-products and individual GARCH$(418, 420)$ processes. Obviously these orders should be taken as upperbounds. For instance there might exist coincidental situations (Granger and Newbold, 1986) in which there exist "quasi" common roots in the determinant and the adjoint (see Nijman and Sentana, 1996 for an example).

A particular case in which there are exact common roots between the implied AR and MA parts is the diagonal model of Bollerslev (1990) where in equation (1) $A_1 = diag(\alpha_{11}, \alpha_{22}, \alpha_{33})$. Hence a diagonal multivariate strong GARCH process is identical to a set of strong GARCH univariate processes with possibly contemporaneous correlated disturbances.[3] Another popular MGARCH model is the BEKK(p, q) of Baba *et al.* (1989). In this specification there are no common roots between the determinant and the adjoint and consequently the general results of propositions 1 and 2 apply.

3 A pure portfolio common GARCH model

Simple multivariate models (e.g. MGARCH(0,1)) do not imply parsimonious low order univariate GARCH processes. Parsimony might be obtained under independence and non-contagion of the volatility. Alternatively, we show that the existence of few factors generating the volatility of asset returns is able to explain those stylized facts. Factor (G)ARCH models have been proposed and extensively studied in the literature, among others by Vrontos, Dellaportas and Politis (2003), Fiorentini, Sentana and Shephard (2004), Lanne and Saikkonen (2007), Hafner and Preminger (2009) and more recently by García-Ferrer, Gonzàlez-Prieto and Peña (2012). We base our results on Cubadda *et al.* (2008, 2009) who derived the conditions under which reduced rank VAR models imply parsimonious marginal ARMA models. This section gives the implications of the presence of one factor generating the volatility observed in financial assets. This will be extended to the k factor case in the next section.

Anticipating the results, it will be shown that the presence of co-movements in the volatility might explain the gap between the theoretical orders and the empirical findings. Let us denote a BEKK(p, q) model with k factors as F-BEKK(p, q, k) (see Lin, 1992) and in particular a F-BEKK$(0, 1, 1)$ such that

$$\begin{aligned} \varepsilon_t &= H_t^{1/2} z_t, \\ H_t &= \Gamma_0 \Gamma_0' + \Gamma_1 \varepsilon_{t-1} \varepsilon_{t-1}' \Gamma_1', \\ &= \Gamma_0 \Gamma_0' + \gamma \varphi' \varepsilon_{t-1} \varepsilon_{t-1}' \varphi \gamma', \end{aligned}$$

where for $k = 1$ we have $rank(\Gamma_1) = rank(\varphi\gamma') = 1$. We can write this system using the half-vec operator $vech$ such that $vech(H_t) = vech(\Gamma_0\Gamma_0') + A_1 vech(\varepsilon_{t-1}\varepsilon_{t-1}')$ or in terms of squared errors and cross products $vech(\varepsilon_t\varepsilon_t') = vech(\Gamma_0\Gamma_0') + A_1 vech(\varepsilon_{t-1}\varepsilon_{t-1}') + v_t$ with the multivariate martingale difference sequence $v_t = vech(\varepsilon_t\varepsilon_t') - vech(H_t)$. $A_1 = L_{N\times n^2}(\Gamma_1 \otimes \Gamma_1) S_{n^2\times N}$ where $L_{N\times n^2}$ is the selection matrix that eliminates the redundant lower-triangular elements of H_t and $S_{n^2\times N}$ selects columnwise the coefficients of the squared returns and sum the columns of the cross-returns (see e.g. Harville, 1997, p

[3] The conditional orthogonal model further assumes that the cross-product term $\alpha_{22} = 0$ (and then $\omega_2 = 0$).

357).[4] A_1 can be written in a reduced form with

$$A_1 = \begin{pmatrix} \gamma_1^2\varphi_1^2 & 2\gamma_1^2\varphi_1\varphi_2 & \gamma_1^2\varphi_2^2 \\ \gamma_1\gamma_2\varphi_1^2 & 2\gamma_1\gamma_2\varphi_1\varphi_2 & \gamma_1\gamma_2\varphi_2^2 \\ \gamma_2^2\varphi_1^2 & 2\gamma_2^2\varphi_1\varphi_2 & \gamma_2^2\varphi_2^2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{\gamma_2}{\gamma_1} \\ (\frac{\gamma_2}{\gamma_1})^2 \end{pmatrix} \begin{pmatrix} \gamma_1^2\varphi_1^2 & 2\gamma_1^2\varphi_1\varphi_2 & \gamma_1^2\varphi_2^2 \end{pmatrix},$$

such that there exists a normalized cofeature matrix

$$\delta = \begin{pmatrix} -\frac{\gamma_2}{\gamma_1} & -\frac{\gamma_2^2}{\gamma_1^2} \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

with $\delta^{'} A = 0_{2\times 3}$. From the determinant $\det(I - AL) = 1 - (\gamma_1\varphi_1 + \gamma_2\varphi_2)^2 L$ and the adjoint (which is not reported to save space) it emerges that all squared returns and their cross-products are ARMA$(1,1)$ and hence we have GARCH(1,1) specifications for the conditional variances and covariances. Importantly enough the combination of the series representing the factor, i.e. $\gamma_1^2\varphi_1^2\varepsilon_{1t-1}^2 + 2\gamma_1^2\varphi_1\varphi_2\varepsilon_{1t-1}\varepsilon_{2t-1} + \gamma_1^2\varphi_2^2\varepsilon_{2t-1}^2$

$$P_{t-1}^2 = (\gamma_1\varphi_1\varepsilon_{1t-1} + \gamma_1\varphi_2\varepsilon_{2t-1})^2,$$

has the same variance as a portfolio made up by the series whose weights are determined by the reduced rank analysis and the factor structure.

Therefore, each volatility and cross-correlation in such a system only depend on that portfolio. This is the reason why we call this model a pure portfolio common GARCH model given in terms of unobserved volatilities

$$\begin{pmatrix} h_{11t} \\ h_{12t} \\ h_{22t} \end{pmatrix} = \begin{pmatrix} \varpi_1 \\ \varpi_2 \\ \varpi_3 \end{pmatrix} + \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} P_{t-1}^2, \tag{6}$$

where the $\beta_i' s$ measure the impact of the volatility of the portfolio on the volatilities and covariances of the underlying assets. Moreover there exist combinations of series such that $\beta_{\perp}' vech(H_t)$ does not depend on the volatility of the portfolio. This approach is also different from the GO-GARCH where the direction for the combination is such that the variance of the returns is maximized using a principal combination analysis. In our case we look at the combinations that are the most correlated with the volatility and covariances of the assets.

Note that this interpretation is made easier using the factor BEKK than from an unrestricted MGARCH with a reduced rank in squared returns and cross-returns. Finding a reduced rank in the general model (1) for instance does not ensure that the coefficient of the cross-correlation is numerically two times the coefficients in front of the returns.

[4]Note that in the one-factor case, A_1 can also be obtained using $\varphi\varphi'(\gamma'\varepsilon_{t-1})^2 = \gamma\varphi'\varepsilon_{t-1}\varepsilon_{t-1}^{'}\varphi\gamma'$. Therefore, the matrix $\varphi\varphi'$ is of rank one as well as the coefficient matrix A_1.

Our findings about the parsimony of the implied series generalize as follows in the s factor case. Note however (see the next section for details) that for $k > 1$ we do not have $rank(A_1) = k$ anymore. Propositions and proofs are given for N but the results are derived in the same manner for n if one does not consider the covariances in the analysis.

Proposition 3 *In a $n-$dimensional F-BEKK$(0, q, k)$, the squared returns and cross-returns have a univariate ARMA(p^*, q^*) representation with the same values for the autoregressive parameters. The orders of p^* and q^* are at most $(N-s)q$. When $k = 1$ and hence $s = N-1$, the orders of p^* and q^* are at most q and hence do not depend on N. As a special case, each component of a multivariate F-BEKK$(0, 1, 1)$ follows a weak GARCH$(1, 1)$ whatever the number of assets jointly considered.*

The propositions and the proofs are applications of the results obtained in Cubadda *et al.* (2009) for the VAR(p). We provide the proof for the $(N-s)$ factor MGARCH$(0, q)$, i.e. the F-BEKK$(0, q, (N-s))$. In this case there exists an $(N \times s)$ full column rank matrix δ (with $N = \frac{n(n+1)}{2}$) such that $\delta^{'} vech(\varepsilon_t \varepsilon_t^{'}) = \delta^{'} v_t$.

Proof. *Let us rewrite* the F-BEKK$(0, q, k)$ *as follows*

$$Q(L)x_t = e_t,$$

where $x_t = Mvech(\varepsilon_t \varepsilon_t')$, $e_t = Mv_t$, $Q(L) = M\Phi(L)M^{-1}$, $M' \equiv [\delta : \delta_\perp]$, $\delta_\perp$ is the orthogonal complement of δ with *span($\delta_\perp$) = span(φ). Given that x_t is a non-singular linear transformation of $vech(\varepsilon_t \varepsilon_t')$, the maximum AR and MA orders of the univariate representation of the elements of $vech(\varepsilon_t \varepsilon_t')$ must be the same as those of the elements of x_t. Since $M^{-1} = [\overline{\delta} : \overline{\delta}_\perp]$, where $\overline{\delta} = \delta(\delta'\delta)^{-1}$,* and $\overline{\delta}_\perp = \delta_\perp(\delta_\perp'\delta_\perp)^{-1}$, *we have*

$$Q(L) = \begin{bmatrix} I_s & 0_{s\times(N-s)} \\ \delta_\perp' \Phi(L)\overline{\delta} & \delta_\perp' \Phi(L)\overline{\delta}_\perp \end{bmatrix},$$

from which it easily follows that $\det[Q(L)] = \det[\delta_\perp' \Phi(L)\overline{\delta}_\perp]$ *is a polynomial of order $(N-s)q$. Hence, the univariate AR order of each element of $vech(\varepsilon_t \varepsilon_t')$ are at most q* when $s = N-1$. *To prove the order of the MA component, let $P(L)$ denote a submatrix of $Q(L)$ that is formed by deleting one of the first s rows and one of the first s columns of $Q(L)$. We can partition $P(L)$ as follows*

$$P(L) = \begin{bmatrix} P_{11} & P_{12} \\ P_{21}(L) & P_{22}(L) \end{bmatrix}. \tag{7}$$

Now, P_{11} is a $(s-1)\times(s-1)$ identity matrix, P_{12} is a $(s-1)\times(N-s)$ matrix of zeros, $P_{21}(L)$ is a $(N-s)\times(s-1)$ polynomial matrix of order q, and $P_{22}(L)$ is a $(N-s)\times(N-s)$ polynomial matrix of order q. Hence, $\det[P(L)] = \det[P_{11}]\det[P_{22}(L)]$, *and therefore* $\det[P(L)]$ *is of order $(N-s)q$. Since cofactors associated with the blocks of $Q(L)$ different from P_{11} are polynomials of degree not larger than*

$(N-s)q$, we conclude that the univariate MA orders of each element of $vech(\varepsilon_t\varepsilon_t')$ are at most $(N-s)q$ and hence q when $s = N-1$ ■

Note again that the link between s and k when $k > 1$ is investigated in the next section.

The above propositions can be easily generalized to the F-BEKK(p, q, k), namely a framework that would include p GARCH coefficient matrices (Lin, 1992) such that

$$H_t = \Gamma_0\Gamma_0' + \Gamma_1\varepsilon_{t-1}\varepsilon_{t-1}^{'}\Gamma_1' + ... + \Gamma_q\varepsilon_{t-q}\varepsilon_{t-q}^{'}\Gamma_q' + G_1H_{t-1}G_1' + ... + G_pH_{t-p}G_p'.$$

If there exists a rank reduction in the Γ_i only (and not in the G_j), each component of the $vech(\varepsilon_t\varepsilon_t')$ follows a weak ARMA($(N-s)\max\{p,q\}$,$(N-s)\max\{p,q\}+p$) as in the general VARMA case with a reduced rank structure (see Proposition 2 for the general case without factors). This means that the moving average part of the ARMA representation is inflated by an additive factor p. Now if the coefficient matrices of the ARCH and the GARCH part share the same left null space, i.e. if there exists a matrix δ such that $\delta'\Gamma_i = \delta' G_j = 0$ for all $i = 1$ to q and for all $j = 1$ to p, the results must be adapted accordingly. In order to show the implications of this modeling, let us consider the adjoint of the block triangular matrices $Q(L)$, $Adj\{Q(L)\} = Q(L)^{-1}\det(Q(L)$ such that

$$Adj\{Q(L)\} = \begin{bmatrix} I_s & 0_{s\times(N-s)} \\ -(\delta_\perp'\Phi(L)\overline{\delta}_\perp)^{-1}\delta_\perp'\Phi(L)\overline{\delta} & (\delta_\perp'\Phi(L)\overline{\delta}_\perp)^{-1} \end{bmatrix} \det[\delta_\perp'\Phi(L)\overline{\delta}_\perp].$$

In the VARMA representation of the *vech* of the BEKK such that

$$Q(L)x_t = MG(L)M^{-1}Mv_t,$$

we have that

$$MG(L)M^{-1} = \begin{bmatrix} I_s & 0_{s\times(N-s)} \\ \delta_\perp' G(L)\overline{\delta} & \delta_\perp' G(L)\overline{\delta}_\perp \end{bmatrix}$$

and consequently

$$
\begin{aligned}
& Adj\{Q(L)\}MG(L)M^{-1} \\
= \; & \det[\delta'_\perp \Phi(L)\overline{\delta}_\perp] \begin{bmatrix} I_s & 0_{s\times(N-s)} \\ -(\delta'_\perp \Phi(L)\overline{\delta}_\perp)^{-1}\delta'_\perp \Phi(L)\overline{\delta} & (\delta'_\perp \Phi(L)\overline{\delta}_\perp)^{-1} \end{bmatrix} \begin{bmatrix} I_s & 0_{s\times(N-s)} \\ \delta'_\perp G(L)\overline{\delta} & \delta'_\perp G(L)\overline{\delta}_\perp \end{bmatrix} \\
= \; & \det[\delta'_\perp \Phi(L)\overline{\delta}_\perp] \begin{bmatrix} I_s & 0_{s\times(N-s)} \\ -(\delta'_\perp \Phi(L)\overline{\delta}_\perp)^{-1}\delta'_\perp \Phi(L)\overline{\delta} + (\delta'_\perp \Phi(L)\overline{\delta}_\perp)^{-1}\delta'_\perp G(L)\overline{\delta} & (\delta'_\perp \Phi(L)\overline{\delta}_\perp)^{-1}\delta'_\perp G(L)\overline{\delta}_\perp \end{bmatrix} \\
= \; & \underbrace{\det[\delta'_\perp \Phi(L)\overline{\delta}_\perp]}_{(N-s)\max(q,p)} \times \\
& \left[\begin{matrix} I_s \\ -\underbrace{(\delta'_\perp \Phi(L)\overline{\delta}_\perp)^{-1}}_{(N-s-1)\max(q,p)-(N-s)\max(q,p)} \underbrace{\delta'_\perp \Phi(L)\overline{\delta}}_{\max(q,p)} + \underbrace{(\delta'_\perp \Phi(L)\overline{\delta}_\perp)^{-1}}_{(N-s-1)\max(q,p)-(N-s)\max(q,p)} \underbrace{\delta'_\perp G(L)\overline{\delta}}_{p} \end{matrix} \right. \\
& \left. \begin{matrix} 0_{s\times(N-s)} \\ \underbrace{(\delta'_\perp \Phi(L)\overline{\delta}_\perp)^{-1}}_{(N-s-1)\max(q,p)-(N-s)\max(q,p)} \underbrace{\delta'_\perp G(L)\overline{\delta}_\perp}_{p} \end{matrix} \right],
\end{aligned}
$$

where the lag polynomial orders are given below each elements in the last expression. If we focus on the maximum order bounds, it turns out that we obtain the final equation representation of orders

$$
ARMA(\overbrace{p^* \leq (N-s)\max\{p,q\}}^{AR \text{ part}}, \overbrace{q^* \leq \max[(N-s)\max\{q,p\}, (N-s-1)\max\{q,p\}+p]}^{MA \text{ part}}).
$$

As an example, a F-BEKK(1,1,1), with the same matrix generating the left null space of the ARCH and the GARCH part, implies univariate GARCH(1,1) models as in the F-BEKK(0,1,1); but GARCH(2,1) models in the absence of the commonality of the left null spaces.

4 Multiple common pure portfolios

The results of the previous sections must be extended to the k factors case with caution. Indeed the presence of k factors implies that the matrices A_i in $(I - A(L))$ are of rank k only in the MGARCH model (1). This is not the case for the BEKK. Let us illustrate this with the BEKK(0,1) $H_t = \Gamma_0\Gamma_0' + \Gamma_1\varepsilon_{t-1}\varepsilon_{t-1}'\Gamma_1'$, where we apply the vectorization operator (see for instance Lütkepohl, 1996) such that

$$
\begin{aligned}
vec(H_t) &= vec(\Gamma_0\Gamma_0') + vec(\Gamma_1\varepsilon_{t-1}\varepsilon_{t-1}'\Gamma_1') \\
&= vec(\Gamma_0\Gamma_0') + (\Gamma_1 \otimes \Gamma_1)vec(\varepsilon_{t-1}\varepsilon_{t-1}'),
\end{aligned}
$$

where $vec(H_t)$ is of size $n^2 \times 1$. Hence when $rank(\Gamma_1) = 1$ we also have that $rank(\Gamma_1 \otimes \Gamma_1) = rank(\Gamma_1) \times rank(\Gamma_1) = 1$. When there are k factors in Γ_1 however, we have that $rank(\Gamma_1) = k$ and $rank(\Gamma_1 \otimes \Gamma_1) =$

k^2. This has some consequences for the model in which one eliminates the redundant lines for the cross-correlations in $vec(H_t)$ as well as for the pure variance model in which we only focus on the vector of variances ε_t^2.

To show the results on the $vech$, let us again consider $L_{N\times n^2}$ the matrix that eliminates the redundant cross-products such that

$$vech(H_t) = L_{N\times n^2}vec(H_t) = L_{N\times n^2}vec(\Gamma_0\Gamma_0') + L_{N\times n^2}(\Gamma_1\otimes\Gamma_1)vec(\varepsilon_{t-1}\varepsilon_{t-1}')$$

and the matrix $S_{n^2\times N}$ that selects the columns corresponding to the variances but sum the columns of $(\Gamma_1\otimes\Gamma_1)$ we obtain

$$vech(H_t) = \gamma_0 + L_{N\times n^2}(\Gamma_1\otimes\Gamma_1)S_{n^2\times N}vech(\varepsilon_{t-1}\varepsilon_{t-1}'), \tag{8}$$

where $\gamma_0 = L_{N\times n^2}vec(\Gamma_0\Gamma_0')$ is a $N-$dimensional vector with the intercepts. It can be shown (see Harville (1997, p 358-9) that

$$rank[L_{N\times n^2}(\Gamma_1\otimes\Gamma_1)S_{n^2\times N}] = \frac{1}{2}[rank(\Gamma_1)]^2 + \frac{1}{2}[rank(\Gamma_1)] = \tilde{k}_{vech} \tag{9}$$

which means that for $k = 1, 2, 3, ...$ in Γ_1, there are respectively $\tilde{k}_{vech} = 1, 3, 6, ...$factors in the $vech$ with reduced rank restrictions. Consequently, one cannot test for $k = 2$ factors in the VAR representation of squared returns and cross-returns with $n = 3$ assets.

Engle and Marcucci (2006) only consider commonalities in volatilities in their pure variance framework. We now study the implications in terms of model representations for the variances if the true model has commonalities in a complete system like in (8).

In order to first investigate the consequences of ignoring the covariances in the left-hand side of (8), let us now define $D_{n\times N}$ a selection matrix that selects the subsystem consisting of the rows of $vech(H_t)$ corresponding to the variances such that $D_{n\times N}vech(H_t) = vecd(H_t) = \varepsilon_t^2$ where $vecd$ denotes the diagonal vectorization operator.

We have that

$$vecd(H_t) = \tilde{\gamma}_0 + D_{n\times N}L_{N\times n^2}(\Gamma_1\otimes\Gamma_1)S_{n^2\times N}vech(\varepsilon_{t-1}\varepsilon_{t-1}'), \tag{10}$$

where $\tilde{\gamma}_0 = D_{n\times N}vec(\Gamma_0\Gamma_0')$. Let us finally assume that there is no omitted variable bias by also excluding the cross-products from the right-side of (10) or that we have a model that projects past squared returns and cross-returns on squared returns (see below). We define $K_{N\times n}$ the matrix that eliminates the columns corresponding to the cross product and have

$$vecd(H_t) = \breve{\gamma}_0 + D_{n\times N}L_{N\times n^2}(\Gamma_1\otimes\Gamma_1)S_{n^2\times N}K_{N\times n}vecd(\varepsilon_{t-1}\varepsilon_{t-1}'). \tag{11}$$

Using the usual inequalities related to the rank of the product of matrices we can easily obtain the ranks for the $vecd$ models in (10) and (11), namely

$$\tilde{k}_{vecd} = \min(n, \tilde{k}_{vech}). \tag{12}$$

Our previous results on the implied univariate models must be adapted using $\tilde{k}_{vech}$ or $\tilde{k}_{vecd}$ instead of k (and therefore $(N-\tilde{k}_{vech})$ and $(n-\tilde{k}_{vecd})$ are now the number of common volatility vectors). For instance, using the same type of proof that we have proposed for the more general model it emerges that in an n-dimensional stationary GARCH$(0,q)$, the individual ARMA processes for the squared excess returns have both AR and MA orders not larger than $\tilde{k}_{vech}q$.

The previous results also show that results obtained by Engle and Marcucci (2006) might be misleading for getting the number of factors k. Indeed, they assume a pure variance specification in which the covariances do not play any role. In practice they consider an exponential form and thus take the log-transform to ensure strictly positive squared returns. However, whether this is a good model description of the data or not, they test for the presence of reduced rank between the squared returns and the past such that

$$\varepsilon_t^2 = \gamma_0 + \varphi\gamma^{'}\varepsilon_{t-1}^2 + v_t,$$

for $\varepsilon_t^2 = (\varepsilon_{1t}^2, ..., \varepsilon_{nt}^2)'$. Consequently if the DGP has a BEKK representation, the numbers that one obtains ($\tilde{k}_{vecd} = \min(n, \tilde{k}_{vech})$) must be translated to get back to k. Finally note that Engle and Marcucci (2006) take the log of the elements of ε_t^2. Furthermore, to avoid taking the log of zero squared returns, they add a tiny constant ι to the squared returns, (i.e. $\ln(\varepsilon_t^2+\iota)$) with the undesirable consequence of introducing large negative values and hence artificially making $\ln(\varepsilon_t^2+\iota)$ closer to an *i.i.d.* process.

Engle and Marcucci (2006) explicitly ignore the cross-products. In our explanation of the model above we have used a matrix $K_{N\times n}$ to get rid of the columns corresponding to the cross-product, assuming that the remaining parameters are unchanged. This can be different when we estimate such a model however. Let us first develop the theoretical model representation of such a misspecified model. We can indeed write the $N-$dimensional MGARCH$(0,q)$ with $N = n(n+1)/2$ such that

$$h_t = \omega + A_q(L)\begin{pmatrix} \varepsilon_t^2 \\ x_t \end{pmatrix}$$

with x_t the vector of $\frac{n(n-1)}{2} = N-n$ cross-product elements $\varepsilon_{it}\varepsilon_{jt}$, $i \neq j$. In terms of observable series we have

$$\{I - A_q(L)\}\begin{pmatrix} \varepsilon_t^2 \\ x_t \end{pmatrix} = \omega + v_t, \qquad v_t = h_t - \begin{pmatrix} \varepsilon_t^2 \\ x_t \end{pmatrix},$$

or more explicitly

$$\begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}\begin{pmatrix} \varepsilon_t^2 \\ x_t \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} + \begin{pmatrix} v_{1t} \\ v_{2t} \end{pmatrix}, \tag{13}$$

where to simplify notation $\phi_{ij} = \phi_{ij}(L)$ are polynomial matrices of degree q in L. Similarly we denote by $|\phi_{ii}|$ and ϕ_{ij}^a respectively the determinant and the adjoint of $\phi_{ij}(L)$. We now marginalize (13) with respect to x_t. This can be done by first computing the final equation representation of the second row of (13) using the identity $|\phi_{22}|I_{N-n} = \phi_{22}^a\phi_{22}$. Next we multiply the first row of (13) by $|\phi_{22}|$ and the new equation for x_t by ϕ_{12}. One obtains the following system of equations

$$\begin{cases} |\phi_{22}|\phi_{11}\varepsilon_t^2 + |\phi_{22}|\phi_{12}x_t = |\phi_{22}|\omega_1 + |\phi_{22}|\upsilon_{1t} \\ \\ \phi_{12}\phi_{22}^a\phi_{21}\varepsilon_t^2 + \phi_{12}|\phi_{22}|x_t = \phi_{12}\phi_{22}^a\omega_2 + \phi_{12}\phi_{22}^a\upsilon_{2t} \end{cases}. \tag{14}$$

Substract the second equation from the first in order to eliminate x_t to get

$$[\underbrace{|\phi_{22}|}_{(N-n)q}\underbrace{\phi_{11}}_{q} - \underbrace{\phi_{12}}_{q}\underbrace{\phi_{22}^a}_{(N-n-1)q}\underbrace{\phi_{21}}_{q}]\varepsilon_t^2 = \omega^* + \underbrace{|\phi_{22}|}_{(N-n)q}\upsilon_{1t} - \underbrace{\phi_{12}}_{q}\underbrace{\phi_{22}^a}_{(N-n-1)q}\upsilon_{2t}, \tag{15}$$

with $\omega^* = |\phi_{22}|\omega_1 - \phi_{12}\phi_{22}^a\omega_2$ and where the maximal polynomial orders are reported beneath each polynomial matrices. This implies an $n-$dimensional VARMA$((N-n)q + q, (N-n)q)$ pure variance process for which the implied maximal univariate orders can be determined using the general rules.

5 Test statistics

In the previous sections we have studied the consequences of the presence of a factor structure in BEKK$(0, q)$. Three factor representations have been derived. We now propose different strategies to detect the presence of these reduced rank structures. A first group of tests building upon the one proposed by Engle and Marcucci (2006) are based on canonical correlation analyses of the VAR representations of models (8), (10) and (11). Then we also consider a standard likelihood ratio test for determining k in a general BEKK(p, q).

As explained above, the first group of tests is inspired by the one proposed by Engle and Marcucci (2006). For $\varepsilon_t = H_t^{1/2}z_t$, the three VAR representations of the generalized versions of models (8), (10) and (11) to the F-BEKK$(0, q, k)$ are as follows

$$\mathcal{M}_1: \quad vech(\varepsilon_t\varepsilon_t^{'}) = \gamma_0 + A_1 vech(\varepsilon_{t-1}\varepsilon_{t-1}^{'}) + ... + A_q vech(\varepsilon_{t-q}\varepsilon_{t-q}^{'}) + v_t,$$

$$\mathcal{M}_2: \quad vecd(\varepsilon_t\varepsilon_t^{'}) = \tilde{\gamma}_0 + \tilde{A}_1 vech(\varepsilon_{t-1}\varepsilon_{t-1}^{'}) + ... + \tilde{A}_q vech(\varepsilon_{t-q}\varepsilon_{t-q}^{'}) + \tilde{v}_t,$$

$$\mathcal{M}_3: \quad vecd(\varepsilon_t\varepsilon_t^{'}) = \breve{\gamma}_0 + \breve{A}_1 vecd(\varepsilon_{t-1}\varepsilon_{t-1}^{'}) + ... + \breve{A}_q vecd(\varepsilon_{t-q}\varepsilon_{t-q}^{'}) + \breve{v}_t,$$

where $v_t, \tilde{v}_t$ are martingale differences and $\breve{v}_t$ is a martingale difference if the DGP is a pure variance model. We aim at determining for $\mathcal{M}_i$, $i = 1, 2, 3$ the number of common factors k driving H_t or similarly the $n - k = s$ vectors annihilating the dynamics. We assume nested reduced rank structures in the A matrices.[5] In order to determine the rank $\tilde{k}_{vech}$ or $\tilde{k}_{vecd}$ corresponding to the number of factors k we first rely on the canonical correlation analysis (as suggested by Engle and Marcucci, 2006 for a specification similar to $\mathcal{M}_3$), i.e. an analysis of the eigenvalues and the eigenvectors of

$$\hat{\Sigma}_{YY}^{-1}\hat{\Sigma}_{YW}\hat{\Sigma}_{WW}^{-1}\hat{\Sigma}_{WY}, \tag{16}$$

or similarly in the symmetric matrix

$$\hat{\Sigma}_{YY}^{-1/2}\hat{\Sigma}_{YW}\hat{\Sigma}_{WW}^{-1}\hat{\Sigma}_{WY}\hat{\Sigma}_{YY}^{-1/2}. \tag{17}$$

$\hat{\Sigma}_{ij}$ are the empirical covariance matrices, Y is the left hand-side variable of one of the three system $\mathcal{M}_i$ and W the right hand-side explanatory variables. For *iid* normally distributed disturbances, the likelihood ratio test statistic for the null hypothesis that there exist at least s linear combinations that annihilate $\tilde{k}_{vecd} = (n-s)$ or $\tilde{k}_{vech} = (N-s)$ features in common to these random variables is given by

$$\zeta_{LR(s)} = -T\sum_{j=1}^{s} \ln(1-\hat{\lambda}_j) \qquad s = 1, \ldots, n \text{ or } N, \tag{18}$$

where $\hat{\lambda}_j$ is the j-th smallest eigenvalue of the estimated matrix (16).[6] In VAR models with Gaussian errors, (18) is the usual likelihood ratio statistics and follows asymptotically a $\chi^2_{(df_i)}$ distribution under the null for $\mathcal{M}_i$, $i = 1, 2, 3$ where respectively $df_1 = sNq - s(N-s)$, $df_2 = snq - s(N-s)$ and $df_3 = snq - s(n-s)$. However in our setting, the errors $v_t, \tilde{v}_t$ and $\breve{v}_t$ are neither *i.i.d.* nor Gaussian but highly skewed and at best martingale difference sequences. Consequently $\zeta_{LR(s)}$ is likely not to be $\chi^2_{(df)}$ distributed under the null. Nevertheless, we consider this case because it is a direct extension of Engle and Marcucci (2006). Note that in their theoretical framework of an exponential pure variance model, the log transformation of the squared residuals first renders the residuals normally distributed (justifying the use (18)) and attenuates the heteroskedasticity of the error term. Whether this test is accurate in a F-BEKK$(0, q, k)$ framework (not imposing an exponential structure and allowing for cross-returns in the factor structure), is an open question that we investigate in the next section by means of a Monte Carlo analysis.

In order to find a multivariate robust counterpart to the canonical correlation approach[7], we propose to modify $\zeta_{LR(s)}$ in two respects. First we use a Wald approach $\zeta_{W(s)}$, asymptotically equivalent to $\zeta_{LR(s)}$

[5] We have also investigated a variant of $\mathcal{M}_2$ with a factor representation on the variances only. To do so, we concentrate out the effect of the covariances before applying $\mathcal{M}_3$ on residuals. The behavior of this test was poor and consequently the results are not reported to save space.

[6] Canonical correlation based tests have been extensively used in economics to test for the presence of common factors and to determine their number (see e.g. Anderson and Vahid, 2007, who proposed a version of this test that is robust to the presence of jumps in the observed series).

[7] In the context of testing for common cyclical features, Candelon *et al.* (2005) have illustrated in a Monte Carlo exercise that $\zeta_{LR(s)}$ has indeed large size distortions in the presence of GARCH disturbances. The solution proposed by the authors was to use a GMM approach in which the covariance matrix is the robust HCSE covariance matrix proposed by White (1982). We do not whish to apply this technique in our paper because it is suited for a single equation approach.

(see Christensen *et al.* 2011). Then we robustify $\zeta_{W(s)}$, a test denoted $\zeta^{rob}_{W(s)}$, using the multivariate extension of the White approach proposed in Ravikumar *et al.* (2000) for system of seemingly unrelated regressions. Indeed because the Wald approach only necessitates the estimation of the unrestricted parameters, we can see models $\mathcal{M}_i$ as SURE systems with heteroskedastic disturbances.

Let us define the reduced rank restrictions, for instance in $\mathcal{M}_1$, as

$$\mathrm{R}vec(A_1 : ...A_q)' = 0_{sd\times 1}, \qquad\qquad \mathrm{R} = \delta' \otimes I_d.$$

R is a $sd \times Nd$ in $\mathcal{M}_1$ and $\mathcal{M}_2$ and R is a $sd \times nd$ in $\mathcal{M}_3$, with d is the number of rows in the rectangular matrix $A = (A_1 : ... : A_q)'$, namely $d = Nq$ for $\mathcal{M}_1$ and $\mathcal{M}_2$ and $d = nq$ for $\mathcal{M}_3$. Using $\hat{\delta}$ obtained by the eigenvectors of the canonical correlation (16), the Wald test is

$$\zeta_{W(s)} = (\mathrm{R}vec\hat{A})'(\mathrm{R}\ \mathrm{Var}(vec\hat{A})\ \mathrm{R}')^{-1}(\mathrm{R}vec\hat{A}),$$

with $\mathrm{Var}(vec\hat{A}) = \hat{V} \otimes (\tilde{W}'\tilde{W})^{-1}$ where $\hat{V}$ is the empirical covariance matrix of the disturbance terms v_t in the unrestricted models and $\tilde{W}$ are the demeaned regressors. $\zeta_{W(s)}$ is asymptotically equivalent to $\zeta_{LR(s)}$ (see Christensen *et al.* 2011). Now in the presence of a time varying multivariate process we compute an estimator of $\mathrm{Var}(vec\hat{A})$ robust to the presence of heteroskedasticity (see Ravikumar, *et al.* 2000) such that

$$\zeta^{rob}_{W(s)} = (\mathrm{R}\ vec\hat{A})'(\mathrm{R}\ \mathrm{robVar}(vec\hat{A})\ \mathrm{R}')^{-1}(\mathrm{R}\ vec\hat{A})$$

where

$$\mathrm{robVar}(\hat{A}) = (I_N \otimes (\tilde{W}'\tilde{W})^{-1})(\sum_{t=1}^{T} \hat{\eta}_t\hat{\eta}_t')(I_N \otimes (\tilde{W}'\tilde{W})^{-1})$$

with

$$\hat{\eta}_t\hat{\eta}_t' = \hat{v}_{.t}\hat{v}_{.t}' \otimes \tilde{W}_{.t}\tilde{W}_{.t}',$$

where $\hat{v}_{.t} = (\hat{v}_{1t}, ...\hat{v}_{Nt})'$ and $\tilde{W}_{.t} = (W_{1t}, ..., W_{dt})$ being the explanatory variables for observations t. $\zeta^{rob}_{W(s)}$ is asymptotically equivalent to $\zeta_{W(s)}$ and hence to $\zeta_{LR(s)}$. Although this robust test is likely to have good properties in the presence of Gaussian but heteroskedastic errors (see Hecq and Issler, 2012), the additional effect of non-normality should deteriorate the performance of the test (as will be illustrated in the next section). For this latter reason and because the above canonical correlation tests are more difficult to implement in a VARMA context[8], i.e. for the general F-BEKK(p, q, k) model, we also propose a standard likelihood ratio test.

The quasi maximum likelihood (QML) estimator is the one that maximizes the Gaussian log likelihood function $L(\vartheta) = \sum_{i=1}^{T} l_t(\vartheta)$ with

$$l_t(\vartheta) = -\frac{n}{2}\ln(2\pi) - \frac{1}{2}\ln|H_t(\vartheta)| - \frac{1}{2}\varepsilon_t' H_t^{-1}(\vartheta)\varepsilon_t.$$

[8] See Tiao and Tsay (1983) for reduced rank analyses in the VARMA(p,q).

Let us denote by respectively $L(\vartheta^{un})$ and $L(\vartheta^{res})$ the likelihood values for the unrestricted full-BEKK and the reduced rank BEKK with k factors. For instance in a BEKK$(0,1)$, the unrestricted model is $H_t = \Gamma_0\Gamma_0' + \Gamma_1\varepsilon_{t-1}\varepsilon_{t-1}'\Gamma_1'$ and the restricted one $H_t = \Gamma_0\Gamma_0' + \alpha\beta_1'\varepsilon_{t-1}\varepsilon_{t-1}'\beta_1\alpha'$. Then the quasi-likelihood ratio is

$$QLR = 2\{L(\vartheta^{un}) - L(\vartheta^{res})\} \sim \chi^2_{(df)}$$

where the number of degrees of freedom is the difference between the number of estimated parameters of the unrestricted and restricted models. As an example, in a BEKK$(0,q)$ this difference is $df = \underbrace{n^2q}_{unrestricted} - \underbrace{(nk + nkq - k^2)}_{restricted}$. The advantage of this QML approach over the canonical correlation is that it can be easily generalized to F-BEKK(p,q,k). From the discussion in Section 3 we favor a model with a same left null space for the ARCH and the GARCH matrix parameters. For instance $H_t = \Gamma_0\Gamma_0' + \alpha\beta_1'\varepsilon_{t-1}\varepsilon_{t-1}'\beta_1\alpha' + \alpha\delta_1'\varepsilon_{t-1}\varepsilon_{t-1}'\delta_1\alpha'$ in a BEKK(1,1,k). The number of degrees of freedom is computed in the same manner. The main drawback of this approach might be the difficulty to estimate the model for large n.

6 Monte Carlo results

We investigate in this section the small sample behavior of $\zeta_{LR(s)}$ and $\zeta^{Rob}_{W(s)}$ for $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ and of QLR. The DGP is a factor BEKK$(0,1,k)$ $H_t = \Gamma_0\Gamma_0' + \Gamma_1\varepsilon_{t-1}\varepsilon_{t-1}'\Gamma_1'$ with $k = rank(\Gamma_1) = 1, 2$ factors for $n = 5$ returns.[9] We take $T = 1000$ and 2000 observations (sample sizes typically encountered with financial time series) and compute the rejection frequencies when testing the null hypothesis at a 5% significance level in 1000 replications. The parameters of the factor BEKK models have been "calibrated" on estimations obtained from some daily observations of stocks used in the empirical application but here we forced the model to have successively $k = 1$ or 2 factors. Table 1 shows that the strategy that consists in estimating directly the unrestricted and the restricted BEKK models, i.e. QLR, is preferred. Indeed, the rejection frequencies for QLR are close to the nominal 5% level.

On the other hand the behavior of the tests statistics $\zeta_{LR(s)}$ and $\zeta^{rob}_{W(s)}$ is not coherent. For instance the canonical correlation approach proposed by Engle and Marcucci (2006) seems to diverge when T increases. While the robust Wald test appears to be adequate with two factors present, it is oversized when testing for one factor; hence numbers around 5% may be accurate but non robust estimates of the true size of the test and therefore only hold for specific DGPs (this has been confirmed on alternative DGPs).

[9] Fewer size distorsions are obtained for smaller n and hence it was less obvious to discriminate between the different approaches. Also note that many alternative strategies have been investigated but are not reported due to their bad performance. For instance a partial least square approach, similar to the one proposed by Cubadda and Hecq (2011) for the VAR, underestimates the number of factors and concludes to multivariate white noise process in most cases. Also, taking the log of the covariance matrix seems to work for some particular cases but not in general.

Table 1: Rejection frequencies of common GARCH factor tests

		$T=1000$			$T=2000$		
		$\zeta_{LR(s)}$	$\zeta^{rob}_{W(s)}$	QLR	$\zeta_{LR(s)}$	$\zeta^{rob}_{W(s)}$	QLR
$\mathbf{k=1}$	BEKK	–	–	7.80	–	–	6.1
	$\mathcal{M}_1$	28.6	100		30.8	98.2	–
	$\mathcal{M}_2$	23.5	44.8		27.5	28.1	–
	$\mathcal{M}_3$	11.5	13.0		13.3	10.1	–
$\mathbf{k=2}$	BEKK	–	–	7.10	–	–	6.1
	$\mathcal{M}_1$	7.5	99.5	–	11.6	90.8	–
	$\mathcal{M}_2$	2.0	4.7	–	3.8	5.8	–
	$\mathcal{M}_3$	0.2	0.5	–	0.1	0.5	–

In order to investigate the local power of this QLR test, we hit the reduced rank matrix Γ_1 such that

$$\Gamma_1^* = \Gamma_1 + \Theta$$

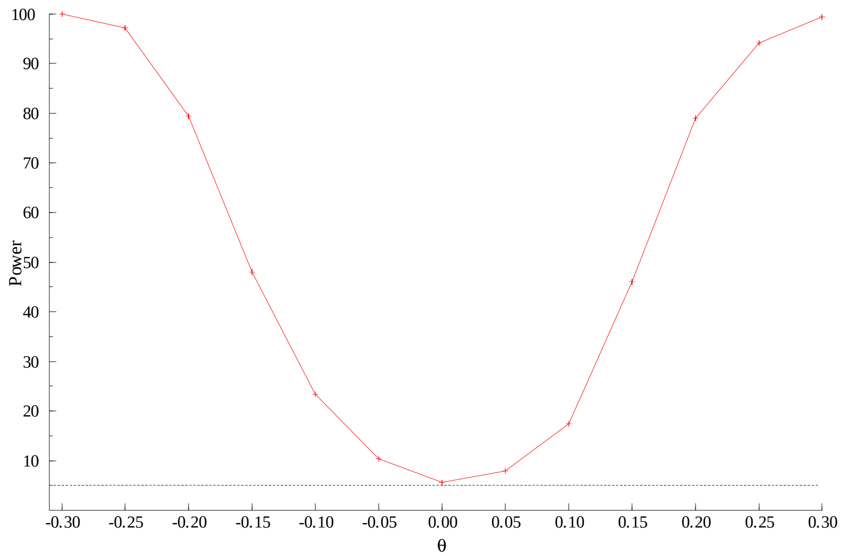
with Θ is a $n \times n$ matrix of zeros but the lower right element which is $\theta = -0.3, -0.25, ..., 0, .., 0.3$. This allows us to present both the unadjusted power when $\theta \neq 0$ and the empirical size when $\theta = 0$ on a same picture. We only consider the case $k = 1$ for $T = 2000$ observations. Again 1000 are used fore each θ. As expected the power quickly converges to 100% the farther away one gets from the null hypothesis.

7 An illustrative example

In this section, the presence of commonalities in volatility within m stock returns is considered. Wrapping up our strategy, we first estimate m univariate models for each of these assets. Then we focus on a group of $n \leq m$ series presenting similarities in terms of parsimony, e.g. GARCH(1,1) specifications. This block of n assets is more deeply studied in a multivariate setting in order to discriminate between a framework with few factors and something else, for instance a diagonal or an orthogonal model. The pure portfolio common volatility model introduced in this paper is a particular factor model generating homoskedastic portfolios for a subset of n out of the m original series (which is by no means riskfree).

To illustrate this approach, the data set we use is obtained from TickData and consists of daily closing transaction prices for fifty large capitalization stocks from the NYSE, AMEX NASDAQ, covering the period from January 1, 1999 to December 31, 2008 (2489 trading days). The appendix provides a list of ticker symbols and company names. For the conditional mean we have estimated AR(2) models with daily dummies to capture Monday and Friday effects. For the conditional variance we have run four different

Figure 1: Power of the quasi-likelihood ratio test



specifications. These are the GARCH(1,1), the GARCH(1,2) and two long memory models, namely FIGARCH$(1, d, 0)$ and FIGARCH$(1, d, 1)$. Out of the 50 series, the GARCH(1,1) model is favoured in six cases using both formal likelihood ratio tests and the Hannan-Quinn information criterion. The six returns with no indication of long memory are (and using their acronyms, see Appendix) ABT, BMY, GE, SLB, XOM, XRX. Table 2 reports the value of the estimated parameters in the conditional variance equation, the results on the conditional mean equations being not reported to save space.

Therefore we consider these six series and we apply our proposed tests. On the six series we consider both a BEKK(0,1) and a BEKK(1,1) where for the latter we impose the same left null space generating the ARCH and the GARCH parts. For these two models we compute the likelihood of the full-BEKK as well as the factor BEKK for $k = 0, 1, 2, 3, 4, 5$; the F-BEKK with $k = 6$ being the unrestricted full-BEKK. Using the QLR test statistics we reject the null of any rank reduction (*p-values* are all smaller than 0.001 and are therefore not reported to save space). Consequently, the parsimony observed for these six returns is more likely due to independent behavior (diagonal or orthogonal model) than to the presence of factors. We also estimated diagonal BEKK(0,1) and a BEKK(1,1) and found that on these 6 series, likelihood ratio tests favour the diagonal models compared to the full-BEKK ones. A diagonal BEKK(1,1) is therefore retained for these series.

Interestingly, we also applied the two reduced rank tests $\zeta_{LR(s)}$ and $\zeta^{rob}_{W(s)}$ in the VAR representation for the squared returns and cross-returns, i.e. $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$. Unlike the QLR test, these tests point out the presence of commonalities in volatility which is very likely due to the strong size distortion of

Table 2: QMLE of GARCH(1,1) on the 6 retained series

	ABT	BMY	GE	SLB	XOM	XRX
ω	0.0139	0.0076	0.0028	0.0284	0.0253	0.0162
	(0.0036)	(0.0027)	(0.0015)	(0.0115)	(0.0071)	(0.0060)
α_1	0.0439	0.0617	0.0433	0.0428	0.0726	0.0612
	(0.0048)	(0.0058)	(0.0049)	(0.0051)	(0.0067)	(0.0053)
β_1	0.9512	0.9405	0.9580	0.9515	0.9180	0.9413
	(0.0054)	(0.0046)	(0.0046)	(0.0065)	(0.0082)	(0.0054)
$Q^2(20)$	0.72	0.96	0.38	0.50	0.28	0.98

Note: Robust standard errors are reported in brackets. $Q^2(20)$ is the p-value of the Ljung-Box test on squared standardized residuals with 20 lags.

these tests highlighted in the previous section.

8 Conclusions

This paper studies the orders of the univariate weak GARCH processes implied by multivariate GARCH models. We recall that except in some coincidental situations, the marginal models are generally non-parsimonious. However, the presence of common features in volatility leads to a large decrease of these implied theoretical orders. We emphasize two radically different structures that may give rise to similar parsimonious univariate representations. These are diagonal models on the one hand, e.g. diagonal-BEKK, and models with reduced rank matrices resulting from the presence of common factors on the other hand, e.g. the F-BEKK.

Consequently we propose different strategies to detect the presence of such GARCH co-movements in a multivariate setting. We believe that this is better than either assuming non-contagion of asset returns from the outset or imposing a factor framework when it is not present. We find that reduced rank test statistics in squared returns (and cross-returns) display severe size distortions while our proposed likelihood ratio test is correctly sized and has good power properties.

Our results plead for looking at individual series prior to a multivariate modelling. For instance, this would help to discover and estimate separately blocks of assets sharing the same sort of dynamic behavior. We won't be able to see it on the whole set of series.

In our application, we detected six series following GARCH(1,1) specifications. We applied our proposed likelihood ratio test and did not find any evidence of commonalities in volatility.

Extensions of this paper are numerous. For instance one could have used bootstrapped versions of some of the tests statistics presented in this paper. This approach has been used by Hafner and Herwartz (2004) to test for causality in the full-BEKK model but with little improvement compared to the asymptotic versions of their test. Next we can further decompose the set of m assets into K blocks with $m = n_1 + ... + n_K$. There might exist a group of series with long memory having similarities in their dynamics.

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10 Stocks used in the empirical application

Symbol	Issue name	Symbol	Issue name
AAPL	APPLE INC	JNJ	JOHNSON &JOHNSON
ABT	ABBOTT LABORATORIES	JPM	JP MORGAN CHASE
AXP	AMERICAN EXPRESS CO	KO	COCA COLA CO
BA	BOEING CO	LLY	ELI LILLY & CO
BAC	BANK OF AMERICA	MCD	MCDONALDS CORP
BMY	BRISTOL MYERS SQ	MMM	3M COMPANY
BP	BP plc	MOT	MOTOROLA
C	CITIGROUP	MRK	MERCK & CO
CAT	CATERPILLAR	MS	MORGAN STANLEY
CL	COLGATE-PALMOLIVE CO	MSFT	MICROSOFT CP
CSCO	CISCO SYSTEMS	ORCL	ORACLE CORP
CVX	CHEVRON CORP	PEP	PEPSICO INC
DELL	DELL INC	PFE	PFIZER INC
DIS	WALT DISNEY CO	PG	PROCTER & GAMBLE
EK	EASTMAN KODAK	QCOM	QUALCOMM
EXC	EXELON CORP	SLB	SCHLUMBERGER N.V.
F	FORD MOTOR CO	T	AT&T CORP
FDX	FEDEX CORP	TWX	TIME WARNER
GE	GENERAL ELEC	UN	UNILEVER N V
GM	GENERAL MOTORS	VZ	VERIZON COMMS
HD	HOME DEPOT INC	WFC	WELLS FARGO & CO
HNZ	H J HEINZ CO	WMT	WAL-MART STORES
HON	HONEYWELL INTL	WYE	WEYERHAEUSER CO
IBM	INTL BUS MACHINE	XOM	EXXON MOBIL
INTC	INTEL CORP	XRX	XEROX CORP